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Direct-Interaction Approximation by the Modal-Interaction Perturbation Technique

Jon Lee[1]

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An alternate method is presented of obtaining the direct-interaction equations by combining the heuristic and rigorous derivations of Kraichnan. Within the framework of the model dynamic representation of Kraichnan's rigorous theory, we have developed the irreducible diagram expansion systematically by formalizing the perturbation argument of his heuristic derivation. It is hoped that the present work will provide a further insight into the analytical structure of the irreducible diagram expansion and bridge the gap apparent in the two original derivations of the direct-interaction equations given by Kraichnan.

KEY WORDS: Kraichnan's direct-interaction approximation; model dynamic representation; irreducible diagram expansion; modal-interaction perturbation technique; Burgers' model turbulence; statistical turbulence theory.

1. PROBLEM STATEMENT

For the covariance level of closure for a turbulent flow problem. Kraichnan[1,2] has presented two distinct—one heuristic and one rigorous—derivations for the so-called direct-interaction (DI) equations. In the earlier, heuristic approach,[1] the DI equations were obtained by computing the first-

[1] Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.

order contribution of the unknown moments in the covariance and averaged response equations from a perturbation induced by the direct-interaction terms. Although this derivation appeals strongly to physical motivation, the importance of this approach is the novel idea of developing perturbation about a certain flow state which is slightly different from the actual turbulent flow. This sort of perturbation will also play a very essential role in our modal-interaction expansion. Kraichnan's later rigorous approach[2] involves imbedding the actual dynamic equation in a model dynamic problem which is constructed by assigning the random coupling coefficients to the nonlinear terms. The essential ingredient of this approach is the irreducible diagram expansion of the unknown moments in the covariance and averaged response equations. Since the random coupling coefficients bring about cancellation of all but the lowest-order irreducible diagram terms upon averaging, the DI equations are obtained as the exact statistical description of the model dynamic problem. Kraichnan has obtained the irreducible diagram expansion by invoking a variational argument which is somewhat intuitive and hence difficult to understand. Furthermore, his variational procedure does not render itself readily to a systematic evaluation of the full irreducible diagram expansion, although this may be an academic question at the present stage of the turbulence theory.

The purpose of this paper is to show that an alternate way of deriving the DI equations is by combining the two distinct methods of Kraichnan that we have just described. Working with the model dynamic representation of his rigourous derivation, we shall show that the irreducible diagram expansion can be developed systematically by formalizing the kind of perturbation argument that Kraichnan has used in his heuristic derivation. This new perturbation procedure will be called the modal-interaction perturbation because it allows us to decompose the dynamic contribution of a certain moment in terms of the modal interactions of various forms. Since the modal-interaction perturbation is prescribed by a formal procedure, we can obtain the irreducible diagram terms of aribtrary order in a completely systematic manner. It is therefore hoped that the present work will provide a further insight into the analytical structure of the irreducible diagram expansion and interrelate the two original derivations of the DI equations given by Kraichnan. Further, we shall state explicitly all the underlying assumptions involved in the DI approximation. In fact, they are neither stronger nor weaker than those invoked implicitly by Kraichnan. In the interest of readers who do not wish to delve into the details, the subsequent sections are subdivided into two parts: the first dealing with the immediate derivation of the DI equations and the second part demonstrating consistency of the irreducible modal-interaction expansions.

As a prototype of the Navier–Stokes equations, the Burgers turbulence equation has the following dynamic model representation using the notations of Ref. 2, Eqs. (11.11)–(11.13):

$$\left[\frac{\partial}{\partial t}\nu - \frac{\partial^2}{\partial x^2} + \bar{u}(x, t)\frac{\partial}{\partial x}\right]\bar{u}(x, t) = -\frac{1}{2}M^{-1}\sum_{\alpha}{}' \frac{\partial}{\partial x}\langle u_\alpha(x, t)\, u_\alpha^*(x, t)\rangle \tag{1}$$

$$\left[\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\bar{u}(x, t)\right] u_\alpha(x, t) = -\frac{1}{2}M^{-1/2}\sum_{\beta}{}'' \phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x} u_\beta(x, t)\, u_{\alpha-\beta}(x, t) \tag{2}$$

$$\left[\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\bar{u}(x, t)\right] G_{\alpha,\gamma}(x, t; x', t') = -M^{-1/2}\sum_{\beta}{}'' \phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x} u_\beta(x, t)\, G_{\alpha-\beta,\gamma}(x, t; x', t') + \delta_{\alpha,\gamma}\,\delta(t - t')\,\delta(x - x') \tag{3}$$

where ν is the kinematic viscosity and $\langle\ \rangle$ denotes the average over an ensemble of M identically and independently distributed realizations. The reality requirement states $u_\alpha(x, t) = u_{-\alpha}^*(x, t)$, where the asterisk denotes the complex conjugate. Note that we omit $\alpha = 0$ in $\sum_\alpha'$ and both $\beta = 0$ and $\beta = \alpha$ in $\sum_\beta''$. The random coupling coefficients $\phi_{\alpha,\beta,\alpha-\beta}$ are assumed to be the same for every realization in the ensemble, and the true problem corresponds to all $\phi = 1$. The ϕ's are subjected to the following dynamic restrictions:

$$\phi_{\alpha,\beta,\alpha-\beta} = \phi_{\alpha,\alpha-\beta,\beta}, \qquad \phi_{-\alpha,-\beta,-\alpha+\beta} = \phi^*_{\alpha,\beta,\alpha-\beta}, \qquad \phi_{\alpha-\beta,-\beta,\alpha} = \phi^*_{\alpha,\beta,\alpha-\beta}.$$

The mean equation (1) and the fluctuation equation (2) are coupled through the Reynolds shear stress and mean-fluctuation terms, and hence they describe the evolution of the mean and fluctuation fields. The response equation (3) describes the perturbation of $u_\alpha(x, t)$ with respect to an infinitesimal disturbance of mode γ introduced at (x', t').

Kraichnan[2] has demonstrated the following statistical properties: For the identically and independently distributed realizations, we have

$$\langle u_\alpha(x, t)\, u_\beta(x', t') \cdots\rangle = 0 \qquad (\alpha + \beta + \cdots \neq 0)$$

and

$$\langle G_{\alpha,\gamma}(x, t; x', t')\rangle = 0 \qquad (\alpha \neq \gamma).$$

Under the additional assumption that the realizations have an initial multivariate Gaussian distribution, the covariance

$$U(x, t; x', t') = \langle u_\alpha(x, t)\, u_\alpha^*(x', t')\rangle$$

and the averaged response function $G(x, t; x', t') = \langle G_{\alpha,\alpha}(x, t; x', t')\rangle$ are independent of α. Furthermore, $G_{\alpha,\alpha}$ is statistically sharp in the limit as $M \to \infty$ and

$$\begin{aligned}&\langle u_\alpha(x, t)\, u_\alpha^*(x', t')\, u_\beta(y, s)\, u_\beta^*(y', s')\rangle \\ &\quad = \langle u_\alpha(x, t)\, u_\alpha^*(x', t')\rangle\langle u_\beta(y, s)\, u_\beta^*(y', s')\rangle + O(M^{-1})\end{aligned}$$

In view of this, the statistical equations of the covariance-level can be formulated from Eqs. (1)–(3):

$$\begin{aligned}&\left[\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} + \bar{u}(x, t) \frac{\partial}{\partial x}\right] \bar{u}(x, t) \\ &\quad = -\frac{1}{2}\frac{\partial}{\partial x} U(x, t; x, t)\end{aligned} \tag{4}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \bar{u}(x, t)\right] U(x, t; x', t') \\ &\quad = S(x, t; x', t')\end{aligned} \tag{5}$$

$$\begin{aligned}&\left[\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \bar{u}(x, t)\right] G(x, t; x', t') \\ &\quad = H(x, t; x', t') + \delta(t - t')\, \delta(x - x')\end{aligned} \tag{6}$$

where

$$\begin{aligned}&S(x, t; x', t') \\ &\quad = -\tfrac{1}{2} M^{-1/2} {\sum_\beta}'' \langle \phi_{\alpha,\beta,\alpha-\beta}(\partial/\partial x)\, u_\beta(x, t)\, u_{\alpha-\beta}(x, t)\, u_\alpha^*(x', t')\rangle\end{aligned} \tag{7}$$

$$\begin{aligned}&H(x, t; x', t') \\ &\quad = -M^{-1/2} {\sum_\beta}'' \langle \phi_{\alpha,\beta,\alpha-\beta}(\partial/\partial x)\, u_\beta(x, t)\, G_{\alpha-\beta,\alpha}(x, t; x', t')\rangle\end{aligned} \tag{8}$$

Equations (4)–(6) are the same as Eqs. (11.21), (11.17), and (11.18) of Kraichnan,[2] who closed them by developing the irreducible diagram expansions for S and H using a variational argument. We shall first obtain in Section 2 the irreducible modal-interaction expansions for u_α and $G_{\alpha,\gamma}$. Such expansions, when introduced into (7) and (8), will give the desired irreducible diagram expansions for S and H, which in turn yield the DI approximation when we randomize the ϕ (Section 3).

2. THE MODAL-INTERACTION PERTURBATION EXPANSIONS

Let us write for simplicity the dynamic equations (2) and (3) in the following forms:

$$LG_{\alpha,\gamma}(\mathbf{x};\mathbf{x}') + \sum_{\sigma}{}'' \Phi_{\alpha,\sigma,\alpha-\sigma} u_\sigma(\mathbf{x})\, G_{\alpha-\sigma,\gamma}(\mathbf{x};\mathbf{x}') = \delta_{\alpha,\gamma}\,\delta(\mathbf{x}-\mathbf{x}') \qquad (9)$$

$$Lu_\alpha(\mathbf{x}) + \tfrac{1}{2}\sum_{\sigma}{}'' \Phi_{\alpha,\sigma,\alpha-\sigma} u_\sigma(\mathbf{x})\, u_{\alpha-\sigma}(\mathbf{x}) = 0 \qquad (10)$$

where $\mathbf{x}$ denotes the space–time vector (x, t),

$$L \equiv (\partial/\partial t) - \nu(\partial^2/\partial x^2) + (\partial/\partial x)\,\bar{u}(x, t),$$

and $\Phi_{\alpha,\beta,\alpha-\beta} = M^{-1/2}\phi_{\alpha,\beta,\alpha-\beta}\,\partial/\partial x$. We shall stipulate the following properties of (9) and (10) as the basic underlying assumptions for the modal-interaction perturbation to be presented:

(A_1) It is assumed that $G_{\alpha,\gamma}(\mathbf{x};\mathbf{x}')$ and $u_\alpha(\mathbf{x})$ are known as the respective solutions of (9) and (10).

(A_2) The $G_{\alpha,\gamma}(\mathbf{x};\mathbf{x}')$ and $u_\alpha(\mathbf{x})$ are affected only infinitesimally when we remove a few product terms from the convolution sums.

The first assumption A_1 seems rather strange at first sight because it presupposes knowledge of the problem solution. This, however, reflects the quirk of modal-interaction expansion in that each term of the expansion expresses an interaction made up of the factors $G_{\alpha,\beta}$ and u_α. Here, we do not look for the perturbation solutions of (9) and (10) in the usual sense. Since there are infinitely many product terms in the convolution sum of (9) and (10), we can justify the second assumption A_2 by noting that each product term of $O(M^{-1/2})$ can influence the $G_{\alpha,\gamma}(\mathbf{x};\mathbf{x}')$ and $u_\alpha(\mathbf{x})$ only infinitesimally in the limit as $M \to \infty$. We shall first develop the modal-interaction expansion for $G_{\alpha,\gamma}(\mathbf{x};\mathbf{x}')$, which is somewhat simpler than that for $u_\alpha(\mathbf{x})$ due to the linearity of (9).

2.1. Response Equation (9)

It is important to observe that the diagonal $G_{\alpha,\alpha}(\mathbf{x};\mathbf{x}')$ describes the propagation of disturbance, whereas the nondiagonal $G_{\alpha,\gamma}(\mathbf{x};\mathbf{x}')$, $\alpha \neq \gamma$, represents the buildup of a phase correlation between the modes α and γ. We see from (9) that the equation for the diagonal $G_{\alpha,\alpha}$ takes the form

$$LG_{\alpha,\alpha}(\mathbf{x};\mathbf{x}') + \sum_{\sigma}{}'' \Phi_{\alpha,\sigma,\alpha-\sigma} u_\sigma(\mathbf{x})\, G_{\alpha-\sigma,\alpha}(\mathbf{x};\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \qquad (11)$$

Let us call (11) the Green's equation so as to distinguish it from the response equation for the nondiagonal $G_{\alpha,\gamma}$. In particular, the response equation for a typical nondiagonal $G_{\alpha-\beta,\alpha}$ becomes

$$LG_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')+\sum_{\sigma}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,G_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}')=0 \tag{12}$$

Our objective here is to decompose the dynamic effect of $G_{\alpha-\beta,\alpha}$ in terms of the modal interactions having different structures. We begin by introducing into (12) the expression

$$G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')=\tilde{G}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')+\Delta G^{[\;\,]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') \tag{13}$$

where $\tilde{G}_{\alpha-\beta,\alpha}$ is a certain state to be specified later, and $\Delta G^{[\;\,]}_{\alpha-\beta,\alpha}$ is the perturbation introduced to satisfy the equality. After rearrangement, we shall put the resulting equation in the form

$$L\,\Delta G^{[\;\,]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')+\sum_{\sigma}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,\Delta G^{[\;\,]}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}')=-\text{RHS} \tag{14}$$

where

$$\text{RHS}=L\tilde{G}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')+\sum_{\sigma}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,\tilde{G}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}') \tag{15}$$

Let us single out a product term for $\sigma=-\beta$ from the sum of (15):

$$\begin{aligned}\text{RHS}=L\tilde{G}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')&+\sum_{\sigma\neq-\beta}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,\tilde{G}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}')\\&+\Phi_{\alpha-\beta,-\beta,\alpha}u_\beta{}^*(\mathbf{x})\,\tilde{G}_{\alpha,\alpha}(\mathbf{x};\mathbf{x}')\end{aligned} \tag{16}$$

Suppose that we define the decomposition (13) uniquely by demanding that the $\tilde{G}_{\alpha-\beta,\alpha}$ shall satisfy

$$L\tilde{G}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')+\sum_{\sigma\neq-\beta}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,\tilde{G}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}')=0 \tag{17}$$

This implies that we are considering perturbation about the state $\tilde{G}_{\alpha-\beta,\alpha}$ which is different from the actual $G_{\alpha-\beta,\alpha}$ only infinitesimally. Since (17) is almost identical to (12) except for a product term ($\sigma=-\beta$) removed from the sum, we may replace $\tilde{G}_{\alpha-\beta,\alpha}$ by $G_{\alpha-\beta,\alpha}$ in the limit as $M\to\infty$ by virtue of A_2. Further, the $G_{\alpha-\beta,\alpha}$ is assumed known by A_1, hence (16) reduces to $\text{RHS}=\Phi_{\alpha-\beta,-\beta,\alpha}u_\beta{}^*G_{\alpha,\alpha}$. Therefore, returning to (14) with this RHS, we find that the perturbation $\Delta G^{[-\beta]}_{\alpha-\beta,\alpha}$ induced by the product term for $\sigma=-\beta$ is described by

$$\begin{aligned}L\,\Delta G^{[-\beta]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')&+\sum_{\sigma}{}''\Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma}u_\sigma(\mathbf{x})\,\Delta G^{[-\beta]}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}')\\&=-\Phi_{\alpha-\beta,-\beta,\alpha}u_\beta{}^*(\mathbf{x})\,G_{\alpha,\alpha}(\mathbf{x};\mathbf{x}')\end{aligned} \tag{18}$$

We note that the Green's equation for (18) is precisely the same as (11), whose solution is assumed known by A_1. Hence the solution of (18) becomes

$$\Delta G^{[-\beta]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') = -\int_{t'}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha-\beta,-\beta,\alpha} u_\beta^*(\mathbf{y})\, G_{\alpha,\alpha}(\mathbf{y};\mathbf{x}') \quad (19)$$

where $\mathbf{y}$ denotes (y, s). Since the factor Φ contains $\partial/\partial y$, its position in the integrand should not be changed arbitrarily. Physically speaking, (19) represents a perturbation of $O(M^{-1/2})$ which is induced by the product term for $\sigma = -\beta$ in the presence of the dynamic interactions of all other product terms. We must point out that (19) has the same form as the variational result [Eq. (4.19) in Ref. 2] of Kraichnan. As the first-order term of the modal-interaction expansion for $G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')$, the perturbation (19) is all that we need to obtain the DI approximation for H. Therefore, those readers interested in the immediate derivation of the DI equations may skip the remainder of this subsection, which demonstrates that (19) is indeed the first-order term of the irreducible modal-interaction expansion.

Suppose that we repeat the perturbation procedure of the previous paragraph by singling out an arbitrary product term $(\sigma = -\beta)$ of (15). Then, the perturbation $G^{[\sigma]}_{\alpha-\beta,\alpha}$ induced by a typical product term, say, for $\sigma = \sigma$, becomes

$$\begin{aligned}&\Delta G^{[\sigma]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')\\ &\quad = -\int_{t'}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma} u_\sigma(\mathbf{y})\, G_{\alpha-\beta-\sigma,\alpha}(\mathbf{y};\mathbf{x}') \quad (20)\end{aligned}$$

which is also of $O(M^{-1/2})$. Since $G_{\alpha-\beta,\alpha}$ represents the buildup of a phase correlation due to the dynamic interactions of all the product terms, we shall express the first-order dynamic contribution by the elementary modal-interactions of the form $\Delta G^{[\ ]}_{\alpha-\beta,\alpha}$; i.e.,

$$G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') = \Delta G^{[-\beta]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') + \sum_{\sigma\neq-\beta}{}'' \Delta G^{[\sigma]}_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') \quad (21)$$

In view of (19), we see that the first $\Delta G^{[-\beta]}_{\alpha-\beta,\alpha}$ in the above involves the diagonal $G_{\alpha,\alpha}$, whereas the $\Delta G^{[\sigma]}_{\alpha-\beta,\alpha}$ of the summand have the nondiagonal $G_{\alpha-\beta-\sigma,\alpha}$, as may be seen from (20). The $G_{\alpha-\beta-\sigma,\alpha}$, however, can further be expressed by an expansion similar to (21):

$$G_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}') = \Delta G^{[-\beta-\sigma]}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}') + \sum_{\mu\neq-\beta-\sigma}{}'' \Delta G^{[\mu]}_{\alpha-\beta-\sigma,\alpha}(\mathbf{x};\mathbf{x}') \quad (22)$$

where the $\Delta G^{[\ ]}_{\alpha-\beta-\sigma,\alpha}$ are obtained from (19) and (20) with $\beta \rightarrow \beta + \sigma$. Similarly, we express the nondiagonal $G_{\alpha-\beta-\sigma-\mu,\alpha}$ which appear in the summand of (22) by

$$G_{\alpha-\beta-\sigma-\mu,\alpha}(\mathbf{x};\mathbf{x}') = \Delta G^{[-\beta-\sigma-\mu]}_{\alpha-\beta-\sigma-\mu,\alpha}(\mathbf{x};\mathbf{x}') + \sum_{\rho\neq-\beta-\sigma-\mu}'' \Delta G^{[\rho]}_{\alpha-\beta-\sigma-\mu,\alpha}(\mathbf{x};\mathbf{x}') \tag{23}$$

where the $\Delta G^{[\ ]}_{\alpha-\beta-\sigma-\mu,\alpha}$ can be written down from (19) and (20) with $\beta \rightarrow \beta + \sigma + \mu$. Thus, upon successively introducing (23) into (22) and the resulting expression into (21), we obtain the modal-interaction expansion for $G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')$ up to the third order:

$$\begin{aligned}
&G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}') \\
&\quad = -\int_{t'}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha-\beta,-\beta,\alpha} u_\beta{}^*(\mathbf{y})\, G_{\alpha,\alpha}(\mathbf{y};\mathbf{x}') \\
&\qquad + \sum_{\sigma\neq-\beta}'' \int_{t'}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma} u_\sigma(\mathbf{y}) \\
&\qquad \times \int_{t'}^{s} ds' \int dy'\, G_{\alpha-\beta-\sigma,\alpha-\beta-\sigma}(\mathbf{y};\mathbf{y}')\, \Phi_{\alpha-\beta-\sigma,-\beta-\sigma,\alpha} u^*_{\beta+\sigma}(\mathbf{y}')\, G_{\alpha,\alpha}(\mathbf{y}';\mathbf{x}') \\
&\qquad - \sum_{\sigma\neq-\beta}'' \sum_{\mu\neq-\beta-\sigma}'' \int_{t'}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha-\beta,\sigma,\alpha-\beta-\sigma} u_\sigma(\mathbf{y}) \\
&\qquad \times \int_{t'}^{s} ds' \int dy'\, G_{\alpha-\beta-\sigma,\alpha-\beta-\sigma}(\mathbf{y};\mathbf{y}')\, \Phi_{\alpha-\beta-\sigma,\mu,\alpha-\beta-\sigma-\mu} u_\mu(\mathbf{y}') \\
&\qquad \times \int_{t'}^{s'} ds'' \int dy''\, G_{\alpha-\beta-\sigma-\mu,\alpha-\beta-\sigma-\mu}(\mathbf{y}';\mathbf{y}'')\, \Phi_{\alpha-\beta-\sigma-\mu,-\beta-\sigma-\mu,\alpha} u^*_{\beta+\sigma+\mu}(\mathbf{y}'') \\
&\qquad \times G_{\alpha,\alpha}(\mathbf{y}'';\mathbf{x}') + \text{higher-order terms}
\end{aligned} \tag{24}$$

We observe that each of the modal-interaction terms in (24) starts with $G_{\alpha-\beta,\alpha-\beta}$ and ends with $G_{\alpha,\alpha}$, and contains an increasing number of the diagonal G's with the intermediate indices. Therefore, the structure of (24) can be demonstrated most simply by the diagrammatics. Let us associate the $\Phi_{\alpha,\beta,\alpha-\beta}$ with an open circle and attach three arrow heads to it. Label each arrow with one of the indices of Φ so that the incoming indices add up to the outgoing indices. Further, we denote the u_α and $G_{\alpha,\alpha}$ by wavy and straight lines, respectively, with the index α. Then, the three modal-interaction terms of (24) can be represented by the diagrams in Fig. 1. Although we have chosen a linear configuration in $G_{\alpha,\alpha}$, the particular configuration is immaterial because the diagrams are a topological representation. To be precise, we must associate the operator

$$\int_{t'}^{t} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x};\mathbf{y})$$

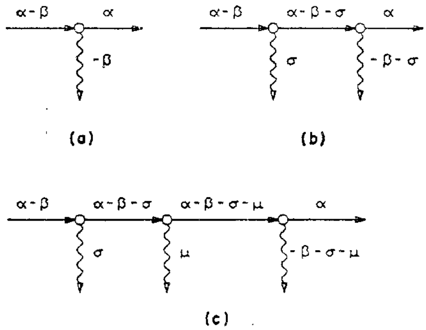


Fig. 1. Diagrams for the modal-interaction terms of $G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')$: (a) first order; (b) second order; (c) third order.

with all the straight lines except the last one in the diagrams; however, this is superfluous for our purpose here. We notice that the diagram of Fig. 1(a) represents the elementary modal-interaction (19). On the other hand, the diagram of Fig. 1(b) is composed of two elementary modal interactions; however, it cannot be split into two separate diagrams of the elementary modal interaction because of the summation restriction $\sigma \neq -\beta$. Similarly, the diagram of Fig. 1(c), consisting of three elementary modal interactions, cannot be reduced to the lower-order diagrams due to the summation restrictions $\sigma \neq -\beta$ and $\mu \neq -\beta - \sigma$. In this respect, the diagrams of Fig. 1 are irreducible and hence (24) is a unique representation of $G_{\alpha-\beta,\alpha}(\mathbf{x};\mathbf{x}')$ in terms of the modal interactions involving only the diagonal G's. In conjunction with the DI approximation, Guderley[3] has first put (9) in a matrix equation form and then expressed the inverse of the matrix in terms of the diagonal elements of the matrix. Upon identifying the diagonal elements of the matrix with the diagonal $G_{\alpha,\alpha}$ and invoking an assumption similar to A_2, we find that (24) has the same structure as his result [Eq. (4.12) of Ref. 3].

2.2. Fluctuation Equation (10)

To develop the modal-interaction expansion for $u_\alpha(\mathbf{x})$, we introduce into (10) the following decomposition:

$$u_\alpha(\mathbf{x}) = \tilde{u}_\alpha(\mathbf{x}) + \Delta u_\alpha(\mathbf{x}) \tag{25}$$

By suppressing the quadratic terms in Δu, the perturbation equation analogous to (14) becomes

$$L\,\Delta u_\alpha^{[\ ]}(\mathbf{x}) + \sum_\sigma{}'' \Phi_{\alpha,\sigma,\alpha-\sigma}\tilde{u}_\sigma(\mathbf{x})\,\Delta u_{\alpha-\sigma}^{[\ ]}(\mathbf{x}) = -\mathrm{RHS} \tag{26}$$

where

$$\mathrm{RHS} = L\tilde{u}_\alpha(\mathbf{x}) + \tfrac{1}{2}\sum_\sigma{}'' \Phi_{\alpha,\sigma,\alpha-\sigma}\tilde{u}_\sigma(\mathbf{x})\,\tilde{u}_{\alpha-\sigma}(\mathbf{x}) \tag{27}$$

Suppose for the moment that the $\tilde{u}_\sigma(\mathbf{x})$ in (26) is replaced by $u_\sigma(\mathbf{x})$. We then see at once that (11) also represents the Green's equation for (26). This is, however, not at all unexpected, because (9) represents a linearized response equation of (10). For the reason to become apparent later in Section 3, we shall call the quadratic terms for $\sigma = \beta$ and $\alpha - \beta$ in the sum of (27) the direct modal pairs and all other quadratic terms ($\sigma \neq \beta$ and $\alpha - \beta$) the indirect modal pairs. In order to describe the perturbation induced by the direct modal pairs, we single out the quadratic terms for $\sigma = \beta$ and $\alpha - \beta$ from the sum of (27). Then, using an argument similar to that used in obtaining (18), we find that the perturbation $\Delta u_\alpha^{[\beta,\alpha-\beta]}$ induced by the direct modal pairs is governed by

$$\begin{aligned} L\,\Delta u_\alpha^{[\beta,\alpha-\beta]}(\mathbf{x}) &+ \sum_\sigma{}'' \Phi_{\alpha,\sigma,\alpha-\sigma}u_\sigma(\mathbf{x})\,\Delta u_{\alpha-\sigma}^{[\beta,\alpha-\beta]}(\mathbf{x}) \\ &= -\Phi_{\alpha,\beta,\alpha-\beta}u_\beta(\mathbf{x})\,u_{\alpha-\beta}(\mathbf{x}) \end{aligned} \tag{28}$$

Since the Green's function for (28) is the solution of (11), which is assumed known by A_1, the solution of (28) becomes

$$\Delta u_\alpha^{[\beta,\alpha-\beta]}(\mathbf{x}) = -\int_{t_0}^{t} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x};\mathbf{y})\,\Phi_{\alpha,\beta,\alpha-\beta}u_\beta(\mathbf{y})\,u_{\alpha-\beta}(\mathbf{y}) \tag{29}$$

This is a perturbation of $O(M^{-1/2})$ induced by the direct modal pairs in the presence of the dynamic interactions of all indirect modal pairs. We shall call $\Delta u_\alpha^{[\beta,\alpha-\beta]}$ the direct pair modal interaction, and it has the same form as the variational result [Eq. (11.26) of Ref. 2] of Kraichnan. Again, we point out that the perturbation (29) is all that is required to obtain the DI approximation for S. In the remainder of this subsection, however, we shall show that (29) is the first-order term of the irreducible modal-interaction expansion for $u_\alpha(\mathbf{x})$.

Repeating the perturbation procedure for (29), we find that the perturbation $\Delta u_\alpha^{[\sigma,\alpha-\sigma]}$ induced by the typical indirect modal pairs, say for $\sigma = \sigma$ and $\alpha - \sigma$, becomes

$$\Delta u_\alpha^{[\sigma,\alpha-\sigma]}(\mathbf{x}) = -\int_{t_0}^{t} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x};\mathbf{y})\,\Phi_{\alpha,\sigma,\alpha-\sigma}u_\sigma(\mathbf{y})\,u_{\alpha-\sigma}(\mathbf{y}) \tag{30}$$

This indirect pair modal interaction is also of $O(M^{-1/2})$. In view of the nonlinear structure of (10), we shall postulate that the dynamic contribution of $u_\alpha(\mathbf{x})$ can be split into a certain noninteractive part $\tilde{u}_\alpha(\mathbf{x})$ and the modal interactions of various forms. Then, for the modal-interaction expansion of first order, we shall take into account the modal interactions of the form $\Delta u_\alpha^{[\quad]}$ which involve only one modal pair, i.e.,

$$u_\alpha(\mathbf{x}) = \bar{u}_\alpha(\mathbf{x}) + \Delta u_\alpha^{[\beta,\alpha-\beta]}(\mathbf{x}) + \sum_{\sigma\neq\beta,\alpha-\beta}'' \Delta u_\alpha^{[\sigma]}(\mathbf{x}) \tag{31}$$

At this stage of expansion, $\bar{u}_\alpha(\mathbf{x})$ would represent a fictitious velocity field from which are removed the dynamic effects of modal pairs. To obtain the second-order modal interactions, we shall sort out from the $\Delta u_\alpha^{[\sigma]}$ a class of the indirect modal pairs that can be linked up to either β or $\alpha - \beta$ by way of an intermediate mode. To carry out this systematically, we first single out the modal pairs $u_\beta u_{\sigma-\beta}$ and $u_{\alpha-\beta}u_{\sigma-\alpha+\beta}$ from the equations for u_σ and obtain, similar to (31),

$$u_\sigma(\mathbf{x}) = \bar{u}_\sigma(\mathbf{x}) + \Delta u_\sigma^{[\beta,\sigma-\beta]}(\mathbf{x}) + \Delta u_\sigma^{[\alpha-\beta,\sigma-\alpha+\beta]}(\mathbf{x}) + \sum_{\substack{\rho\neq\beta,\sigma-\beta\\ \neq\alpha-\beta,\sigma-\alpha-\beta}}'' \Delta u^{[\rho]}(\mathbf{x}) \tag{32}$$

Then, singling out the modal pairs $u_\beta u_{\alpha-\beta-\sigma}$ and $u_{\alpha-\beta}u_{\beta-\sigma}$ from the equations for $u_{\alpha-\sigma}$, we obtain

$$u_{\alpha-\sigma}(\mathbf{x}) = \bar{u}_{\alpha-\sigma}(\mathbf{x}) + \Delta u_{\alpha-\sigma}^{[\beta,\alpha-\beta-\sigma]}(\mathbf{x}) + \Delta u_{\alpha-\sigma}^{[\alpha-\beta,\beta-\sigma]}(\mathbf{x}) + \sum_{\substack{\mu\neq\beta,\alpha-\beta-\sigma\\ \neq\alpha-\beta,\beta-\sigma}}'' \Delta u_{\alpha-\sigma}^{[\mu]}(\mathbf{x}) \tag{33}$$

Here, the $\Delta u^{[\quad]}$ can be written down explicitly from (29) and (30) with a proper interchange of indices. Introduction of (32) and (33) into (31) gives the leading terms in the modal-interaction expansion for $u_\alpha(\mathbf{x})$:

$$u_\alpha(\mathbf{x}) = \bar{u}_\alpha(\mathbf{x}) - \int_{t_0}^{t} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha,\beta,\alpha-\beta} u_\beta(\mathbf{y})\, u_{\alpha-\beta}(\mathbf{y})$$

(A)

$$- \sum_{\sigma\neq\beta,\alpha-\beta}'' \int_{t_0}^{t} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x};\mathbf{y})\, \Phi_{\alpha,\sigma,\alpha-\sigma} \{\bar{u}_\sigma(\mathbf{y})\, \bar{u}_{\alpha-\sigma}(\mathbf{y})$$

(B)

$$- \bar{u}_\sigma(\mathbf{y}) \int_{t_0}^{s} ds' \int dy'\, G_{\alpha-\sigma,\alpha-\sigma}(\mathbf{y};\mathbf{y}')$$

$$\times\ [\Phi_{\alpha-\sigma,\beta,\alpha-\sigma-\beta} u_\beta(\mathbf{y}')\, u_{\alpha-\beta-\sigma}(\mathbf{y}') + \beta \to \alpha - \beta]$$

(C)

$$- \bar{u}_{\alpha-\sigma}(\mathbf{y}) \int_{t_0}^{s} ds' \int dy'\, G_{\sigma,\sigma}(\mathbf{y};\mathbf{y}')$$

$$\times\ [\Phi_{\sigma,\beta,\sigma-\beta} u_\beta(\mathbf{y}')\, u_{\sigma-\beta}(\mathbf{y}') + \beta \to \alpha - \beta]$$

(D)

$$+ \int_{t_0}^{s} ds' \int dy' \, G_{\sigma,\sigma}(\mathbf{y}; \mathbf{y}')[\Phi_{\sigma,\beta,\sigma-\beta} u_\beta(\mathbf{y}') \, u_{\sigma-\beta}(\mathbf{y}') + \beta \to \alpha - \beta]$$

$$\times \int_{t_0}^{s} ds'' \int dy'' \, G_{\alpha-\sigma,\alpha-\sigma}(\mathbf{y}; \mathbf{y}'')$$

$$\times \underbrace{[\Phi_{\alpha-\sigma,\beta,\alpha-\beta-\sigma} u_\beta(\mathbf{y}'') \, u_{\alpha-\beta-\sigma}(\mathbf{y}'') + \beta \to \alpha - \beta]}_{(\mathrm{E})}$$

$$+ \text{higher-order terms}\Big\} \tag{34}$$

where the second term in each square bracket can be written out explicitly by using $\beta \to \alpha - \beta$ in the first term.

For easier reference to the modal-interaction terms of (34), we have labeled them alphabetically. Since the term (B) cannot induce dynamic interaction with either β or $\alpha - \beta$, it will be dropped from the further discussion. Here, again, the structure of (34) will be examined by the diagrammatics. In addition to the diagrammatic notations of Section 2.1, we shall denote $\bar{u}_\alpha$ by the same wavy line as u_α. Then, the direct pair modal interaction (A) can be represented by the treelike (logged down) diagram of Fig. 2(a). The next higher-order modal-interaction terms (C) and (*D*), which all contain a factor $\bar{u}$, are represented by the typical diagram of Fig. 2(b). Finally, the modal-interaction (E) has the diagrammatic representation of Fig. 2(c). Since the diagrams of Figs. 2(b) and 2(c) cannot be broken down into the

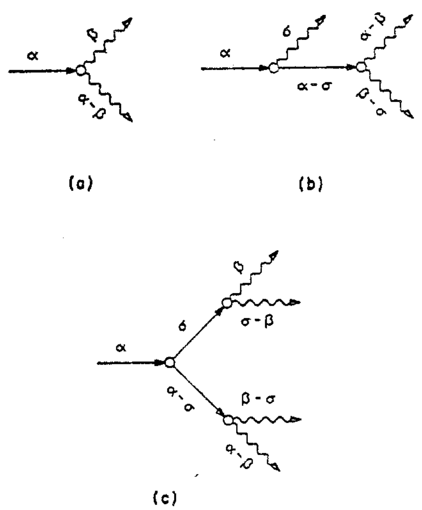


Fig. 2. Diagrams for the modal-interaction terms of $u_\alpha(\mathbf{x})$: (a) first order; (b) second order; (c) third order.

lower-order ones, due to the summation restrictions, we can claim (34) to be the irreducible diagram representation.

3. THE DIRECT-INTERACTION APPROXIMATION

By introducing the modal-interaction expansions of Section 2 into (7) and (8), we can formally obtain the irreducible diagram expansions for $S(\mathbf{x}; \mathbf{x}')$ and $H(\mathbf{x}; \mathbf{x}')$ in a straightforward manner. Since the random coupling coefficients result in cancellation of all but the lowest-order irreduble diagram terms, it suffices to consider only the first-order modal-interaction terms (19) and (29) for the DI approximation. However, in order to exhibit consistency of the irreducible diagram expansions, we shall show that certain higher-order terms of $S(\mathbf{x}; \mathbf{x}')$ and $H(\mathbf{x}; \mathbf{x}')$ under all $\phi = 1$ can be compared with the results obtained previously by other theories[2,4,5]

$H(x, t; x', t')$. Let us introduce into (8) the modal-interaction expansion for $G_{\alpha-\beta,\alpha}$, (19) or (24). We then find that the leading term of $H(\mathbf{x}; \mathbf{x}')$ becomes

$$\begin{aligned} H(x, t; x', t') &= C\left\langle \int_{t'}^{t} ds \int dy \frac{\partial}{\partial x} G_{\alpha-\beta,\alpha-\beta}(\mathbf{x}; \mathbf{y}) \frac{\partial}{\partial y} [u_\beta(\mathbf{x})\, u_\beta^*(\mathbf{y})\, G_{\alpha,\alpha}(\mathbf{y}; \mathbf{x}')] \right\rangle \\ &\quad + \text{higher-order terms of } O(\phi^3) \end{aligned} \tag{35}$$

where

$$C = M^{-1} \sum_{\beta}'' \phi_{\alpha,\beta,\alpha-\beta} \phi_{\alpha-\beta,-\beta,\alpha} .$$

We shall introduce the further assumptions[2] that (i) the ensemble average commutes with the integral and differential operations, (ii) due to the statistical sharpness of $G_{\alpha,\alpha}$,

$$\begin{aligned} &\langle G_{\alpha-\beta,\alpha-\beta}(\mathbf{x}; \mathbf{y})\, u_\beta(\mathbf{x})\, u_\beta^*(\mathbf{y})\, G_{\alpha,\alpha}(\mathbf{y}; \mathbf{x}')\rangle \\ &\quad \to \langle G_{\alpha-\beta,\alpha-\beta}(\mathbf{x}; \mathbf{y})\rangle \langle u_\beta(\mathbf{x})\, u_\beta^*(\mathbf{y})\rangle \langle G_{\alpha,\alpha}(\mathbf{y}; \mathbf{x}')\rangle \end{aligned}$$

and (iii) $C = 1$ under the random coupling model. Orszag[6] has criticized the assumption (ii) on the ground that it is responsible for the violation of Galilean invariance by the DI equations. Upon randomizing the phase of ϕ, the higher-order terms not shown explicitly in (35) all drop out in the limit as $M \to \infty$. Hence, in view of the statistical properties mentioned above and in Section 1, Eq. (35) takes the following form without approximation:

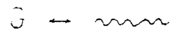
$$H(x, t; x', t') = \frac{\partial}{\partial x} \int_{t'}^{t} ds \int dy\, G(\mathbf{x}; \mathbf{y}) \frac{\partial}{\partial y} [U(\mathbf{x}; \mathbf{y})\, G(\mathbf{y}; \mathbf{x}')] \tag{36}$$

This is the DI approximation, which agrees with the result [Eq. (11.23) in Ref. 2] of Kraichnan.

To exhibit the consistency of the irreducible diagram expansion, let us examine the next higher-order term of (36) which survives under the special condition that all $\phi = 1$ and the u_α have a Gaussian distribution. Instead of writing out the complicated expression in detail, we shall contend here with the examination of the skeleton structure of such a higher-order term by the diagrammatics. To this end, we drop all the arrows and indices from the diagram of Fig. 1(c). Combining such an abridged diagram with

which represents the factor $\Phi_{\alpha,\beta,\alpha-\beta} u_\beta(\mathbf{x})$ in (8), we find that the irreducible configuration upon pairing the wavy lines is given by Fig. 3, in which

and



It can be shown that the diagram of Fig. 3 corresponds to the analytical expression of Kraichnan's inadmissible higher approximation (Section 6 of Ref. 2). Furthermore, it reduces to a first-order term of the G-expansion obtained by Lee (Fig. 15 of Ref. 4) upon introducing the modified vertex operator.

S(x, t; x′, t′). Since the triple moment $S(\mathbf{x}; \mathbf{x}')$ is made up of the triad modal-interactions, the irreducible diagram expansion of (7) involves three modal-interaction expansions for u_α^*, u_β, and $u_{\alpha-\beta}$, respectively. Let us first develop the modal-interaction expansion for u_α^* by singling out the direct modal pairs $2u_\beta^* u_{\alpha-\beta}^*$, which becomes, in view of (29) or (34),

$$u_\alpha^*(\mathbf{x}') = \bar{u}_\alpha^*(\mathbf{x}') - \int_{t_0}^{t'} ds \int dy\, G_{\alpha,\alpha}(\mathbf{x}'; \mathbf{y})\, \Phi^*_{\alpha,\beta,\alpha-\beta} u_\beta^*(\mathbf{y})\, u^*_{\alpha-\beta}(\mathbf{y}) + \cdots \tag{37}$$

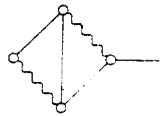


Fig. 3. Abridged diagram for the second-order term of $H(x, t; x', t')$.

Similarly, by singling out the direct modal pairs $2u_\alpha u^*_{\alpha-\beta}$ and $2u_\alpha u_\beta^*$, the modal-interaction expansions for u_β and $u_{\alpha-\beta}$ become, respectively,

$$u_\beta(\mathbf{x}) = \bar{u}_\beta(\mathbf{x}) - \int_{t_0}^{t} ds \int dy\, G_{\beta,\beta}(\mathbf{x}\,;\mathbf{y})\, \Phi_{\beta,\alpha,\beta-\alpha} u_\alpha(\mathbf{y})\, u^*_{\alpha-\beta}(\mathbf{y}) + \cdots \tag{38}$$

$$u_{\alpha-\beta}(\mathbf{x}) = \bar{u}_{\alpha-\beta}(\mathbf{x}) - \int_{t_0}^{t} ds \int dy\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x}\,;\mathbf{y})\, \Phi_{\alpha-\beta,\alpha,-\beta} u_\alpha(\mathbf{y})\, u_\beta^*(\mathbf{y}) + \cdots \tag{39}$$

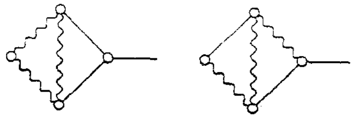
The three dots in the above represent the higher-order terms similar to those of (34). Introducing (37)–(39) into (7), we find that the lower-order terms of $S(\mathbf{x}; \mathbf{x}')$ become

$$\begin{aligned} S(x, t; x', t') &= -\frac{1}{2} M^{-1/2} \sum_\beta{}'' \left\langle \phi_{\alpha,\beta,\alpha-\beta} \frac{\partial}{\partial x} \bar{u}_\beta(\mathbf{x})\, \bar{u}_{\alpha-\beta}(\mathbf{x})\, \bar{u}_\alpha{}^*(\mathbf{x}') \right\rangle \\ &+ \frac{C}{2} \left\langle \int_{t_0}^{t} ds \int dy \frac{\partial}{\partial x} G_{\beta,\beta}(\mathbf{x}; \mathbf{y}) \frac{\partial}{\partial y} [u_\alpha(\mathbf{y})\, \bar{u}_\alpha{}^*(\mathbf{x}')\, u^*_{\alpha-\beta}(\mathbf{y})\, \bar{u}_{\alpha-\beta}(\mathbf{x})] \right\rangle \\ &+ \frac{C}{2} \left\langle \int_{t_0}^{t} ds \int dy \frac{\partial}{\partial x} G_{\alpha-\beta,\alpha-\beta}(\mathbf{x}; \mathbf{y}) \frac{\partial}{\partial y} [u_\alpha(\mathbf{y})\, \bar{u}_\alpha{}^*(\mathbf{x}')\, u_\beta{}^*(\mathbf{y})\, \bar{u}_\beta(\mathbf{x})] \right\rangle \\ &+ \frac{C}{2} \left\langle \int_{t_0}^{t} ds \int dy \frac{\partial}{\partial x} G_{\alpha,\alpha}(\mathbf{x}'; \mathbf{y}) \frac{\partial}{\partial y} [u_\beta{}^*(\mathbf{y})\, \bar{u}_\beta(\mathbf{x})\, u^*_{\alpha-\beta}(\mathbf{y})\, \bar{u}_{\alpha-\beta}(\mathbf{x})] \right\rangle \\ &+ \text{higher-order terms of } O(\phi^3) \end{aligned} \tag{40}$$

where C is defined as in (35). The dynamic significance of (40) is that it describes buildup of the triple moment in terms of the modal interactions having different structures. Since the first term is the triple moment in a fictitious field of no triad modal interaction, we may identify it with the initial value of the triple moment. Then, it assumes zero value under the initial Gaussian condition. By invoking A_2, we can justify $\langle \bar{u}_\alpha u_\alpha{}^* \rangle \to \langle u_\alpha u_\alpha{}^* \rangle$ in the limit as $M \to \infty$. Using the statistical assumptions mentioned previously, we find that (40) reduces to give

$$\begin{aligned} S(x, t\,; x', t') = \int_{t_0}^{t} ds \int dy \frac{\partial}{\partial x} G(\mathbf{x}\,; \mathbf{y}) \frac{\partial}{\partial y} [U(\mathbf{x}\,; \mathbf{y})\, U(\mathbf{x}'\,; \mathbf{y})] \\ + \frac{1}{2} \int_{t_0}^{t'} ds \int dy \frac{\partial}{\partial x} G(\mathbf{x}'\,; \mathbf{y}) \frac{\partial}{\partial y} [U(\mathbf{x}\,; \mathbf{y})]^2 \end{aligned} \tag{41}$$

This is the DI approximation, which agrees with the result [Eq. (11.12) in Ref. 2] of Kraichnan. Note that (41) is the exact result, for the higher-order terms not shown explicitly in (40) would make no contribution under the random coupling model.

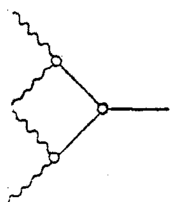


(a) (b)

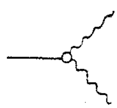
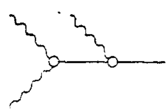
Fig. 4. Abridged diagrams for the two second-order terms of $S(x, t; x', t')$.

Before closing, we shall briefly examine the next higher-order terms of (41) which survive under the same condition as in the discussion of $H(\mathbf{x}; \mathbf{x}')$. Using the abridged diagram representation, the skeleton structure of the two second-order terms can be demonstrated by Fig. 4. The first diagram (Fig. 4a) is made up of two

and a



We see that it is included in Wyld's first-order term for the U-expansion (Fig. 5 of Ref. 5), when the modified vertex operator is introduced. On the other hand, the second diagram (Fig. 4b), consisting of

 ,  , and 

is included in Wyld's second-order term.

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